

Single-Peaked Consistency for Weak Orders Is Easy

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Abstract

It is well known that single-peaked consistency for total orders is in P. However in practice a preference profile is not always comprised of total orders. Often voters have indifference between some of the candidates. In a weak preference order indifference must be transitive. We show that single-peaked consistency for weak orders is in P for the three different variants of single-peakedness for weak orders. Specifically, we consider Black’s original definition for weak orders (Black 1948), single-plateaued preferences (Black 1958), and the existential model recently introduced by Lackner (2014). We accomplish this by transforming each of these single-peaked consistency problems to the problem of determining if a 0-1 matrix has the property of consecutive ones in rows.

1 Introduction

Single-peakedness is one of the most important and commonly examined domain restrictions on preferences. The study of single-peaked preferences in computational social choice is often restricted to total orders, but in practical settings voters often have some degree of indifference in their preferences. This is readily seen in the widely available preference datasets in PREFLIB, which contains several large datasets comprised of voters with partial preferences, many of which are weak orders (Mattei and Walsh 2013). Some election systems are even explicitly defined for weak orders, e.g., the Kemeny rule (Kemeny 1959).

Single-peaked preferences model the preferences of a collection of voters with respect to a one-dimensional axis (Black 1948). Each voter in a single-peaked electorate has a single most preferred candidate (their peak) on the axis and candidates further to the left or right from their peak with respect to the axis are less preferred. This models an election with a single polarizing issue where candidates representing the two extremes would appear on the leftmost and rightmost locations on the axis. Single-plateaued preferences model the preferences of a collection of voters in a similar way, but allow voters to have multiple most preferred candidates (Black 1958).

Elections where the voters have single-peaked preferences over the candidates have many desirable properties. For example, when the voters in an election have single-peaked (or even *nearly* single-peaked) preferences the complexity of determining if a manipulative action ex-

ists often becomes easy (Faliszewski et al. 2011; Faliszewski, Hemaspaandra, and Hemaspaandra 2014) and determining the winner for Dodgson and Kemeny elections becomes easy (Brandt et al. 2010) when it is Θ_2^P -complete in general (Hemaspaandra, Hemaspaandra, and Rothe 1997; Hemaspaandra, Spakowski, and Vogel 2005).

The first paper to computationally study the single-peaked consistency for partial preferences was Lackner (2013), where a partial order is said to be single-peaked with respect to an axis A if it can be extended to a total order that is single-peaked with respect to A . For clarity we refer to this as existentially single-peaked, or single-peaked $_{\exists}$, throughout this paper. Lackner presents algorithms and complexity results for determining the single-peaked consistency for preference profiles of varying degrees of partial preference, including top orders, weak orders, local weak orders, and partial orders in his model (Lackner 2013; 2014). Lackner (2014) shows that if a given preference profile contains an implicitly specified total order, which is not guaranteed to exist, then single-peaked $_{\exists}$ consistency for weak orders and the general case of single-peaked $_{\exists}$ consistency for top orders (weak orders with all indifferent candidates ranked last) are both in P. The complexity of the general case of determining the single-peaked $_{\exists}$ consistency for weak orders is explicitly left as the main open problem in Lackner (2014) and we show that it is in P.

We show that an algorithm to determine if a 0-1 matrix has the property of consecutive ones in rows, previously used to determine the single-peaked consistency for total orders by Bartholdi and Trick (1986) and the single-crossing consistency for total orders by Bredereck et al. (2013) can be used to determine the single-peaked consistency, single-plateaued consistency, and single-peaked $_{\exists}$ consistency for weak orders *without* an implicitly specified total order. So given a preference profile of weak orders not only can we determine if it is single-peaked, single-plateaued, or single-peaked $_{\exists}$, we can find all consistent axes through the use of the PQ-tree algorithm for the consecutive-ones matrix problem (Booth and Lueker 1976).

This paper is organized as follows. In Section 2 we define the types of partial preference studied, state definitions for the different variants of single-peakedness, and define the consecutive-ones matrix problem. Our results are split into three different sections each one corresponding to a differ-

ent variant of single-peakedness. Specifically Section 3 for single-peaked_∃ preferences, Section 4 for single-plateaued preferences, and Section 5 for single-peaked preferences. In each of these sections we redefine the variant of single-peakedness using forbidden substructures and describe the transformation from its consistency problem to the problem of determining if a 0-1 matrix has the property of consecutive ones in rows. In Section 6 we discuss related work that considers single-peakedness and partial preferences. We conclude in Section 7 by summarizing our results and stating some possible directions for further study.

2 Preliminaries

A *preference order*, v , is an ordering of a finite set of candidates, C . A multiset of preference orders is called a *preference profile*, V . (We sometimes refer to each v as a voter with a corresponding preference order.) A *partial order* is a transitive, reflexive, and antisymmetric binary relation on a set. A *weak order* is a partial order where the indifference relation is transitive (see, e.g., (Lackner 2014)), and a *top order* is a weak order where all candidates ranked indifferent are ranked last, and a *total order* is a partial order with no indifference between candidates.

Example 1 Given the set of alternatives $\{a, b, c, d\}$, an example of a total order is $(a > b > d > c)$, an example of a weak order is $(a \sim c > d > b)$, and an example of a top order is $(a > c > b \sim d)$, where “ \sim ” is used to denote indifference.

We focus on weak orders since they model easily understood partiality in preferences. Voters are often not able to discern between two candidates or even view them as truly equal. Allowing each voter to state a weak preference order still requires that they have each candidate in a position in their order, but gives them the ability to have multiple candidates at each position.

It is very natural for election systems to be defined on weak orders. The Kemeny rule is defined on weak orders (Kemeny 1959) and clearly election systems based on pairwise comparisons (e.g., Copeland) can also be evaluated on partial votes. The Borda count can be defined on top orders (Emerson 2013) and a recent paper has even explored the complexity of manipulation on such variants of the Borda count (Narodytska and Walsh 2014).

2.1 Variants of Single-Peakedness

In our definitions of each variant of single-peakedness we refer to the total ordering that each preference profile is consistent with as the axis A .

We begin our discussion of single-peaked preferences by stating the definition for single-peaked preferences for total orders. We use the definition found in Bartholdi and Trick (1986).

Definition 2 A preference profile V of total orders is single-peaked if there exists an axis A such that for every preference order $v \in V$, A can be split at the top preferred candidate (the peak) of v into two segments (one of which may be

empty) X and Y such that v has strictly increasing preference along X and v has strictly decreasing preference along Y .

We now define each of the three variants of single-peakedness for weak orders.

Single-Peaked Preferences. Single-peaked preferences on weak orders can be defined in exactly the same way as single-peaked preferences on total orders.

Definition 3 A preference profile V of weak orders is single-peaked if there exists an axis A such that for every preference order $v \in V$, A can be split at the top preferred candidate (the peak) of v into two segments (one of which may be empty) X and Y such that v has strictly increasing preference along X and v has strictly decreasing preference along Y .

Notice that for a weak preference order to be single-peaked it can only contain indifference between at most two candidates at each position. Otherwise the sets X and Y referred to in Definition 3 would not be strictly increasing/decreasing.

Single-Plateaued Preferences. A slightly weaker restriction than single-peaked preferences on weak orders (that still coincides with single-peaked preferences on total orders) are single-plateaued preferences (Black 1958, Chapter 5) which allow a preference order to have multiple best options (an indifference plateau) instead of a single most preferred candidate (a peak), but otherwise be single-peaked. Building upon the above definition for single-peaked preferences, we state a definition for single-plateaued preferences.

Definition 4 A preference profile V of weak orders is single-plateaued if there exists an axis A such that for every preference order $v \in V$, A can be split into three segments X , Y , and Z where v 's top preferred candidates are all in Y , v has strictly increasing preference along X , and v has strictly decreasing preference along Z and both X and Z can each be empty.

Existentially Single-Peaked Preferences. Single-peaked consistency for partial orders can also be defined through extensions to total orders.

Definition 5 (Lackner 2014) A preference profile V of partial orders is single-peaked_∃ with respect to an axis A if for every $v \in V$, v can be extended to a total order v' such that the preference profile V' of total orders is single-peaked with respect to A .

We can restate single-peaked_∃ preferences for weak orders without referring to extensions to better see how it relates to single-peaked and single-plateaued preferences.

Definition 6 A preference profile V of weak orders is single-peaked_∃ if there exists an axis A such that for every preference order $v \in V$, A can be split into three segments X , Y , and Z where v 's top preferred candidates are all in Y , v has increasing preference along X , and v has decreasing preference along Z and both X and Z can each be empty.

Figure 1 illustrates an example of each variant of single-peakedness for weak orders described where each preference order's consistent axis is $A = a < d < b < e < c$.

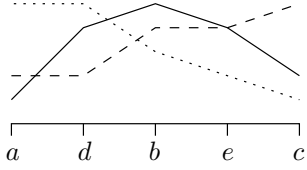


Figure 1: The solid line represents the single-peaked preference order $b > d \sim e > c > a$, the dotted line represents the single-plateaued preference order $a \sim d > b > e > c$, and the dashed line represents the single-peaked₃ preference order $c > b \sim e > d \sim a$.

In Figure 1 the preference order $b < d \sim e > c > a$ is single-peaked, single-plateaued, and single-peaked₃ with respect to A . The preference order $a \sim d > b > e > c$ is single-plateaued and single-peaked₃ with respect to A but not single-peaked since it has more than one most preferred candidate. The preference order $c > b \sim e > d \sim a$ is only single-peaked₃ with respect to A but not single-plateaued or single-peaked since it is not strictly decreasing from its most preferred candidate.

We conclude the discussion of the variants of single-peakedness for weak orders by showing that a single-peaked₃ consistent preference profile does not have a transitive majority relation. Note that single-peaked preferences have a transitive majority relation in general (Black 1948) and single-plateaued preferences have a transitive majority relation when the number of voters is odd (Fishburn 1973).

Consider the preference profile V comprised of the following five voters.¹

$$\begin{array}{ll} v_1 & b > a > c \\ v_2, v_3 & c > b > a \\ v_4, v_5 & a > b \sim c \end{array}$$

Clearly V is single-peaked₃ consistent with the axis $A = a < b < c$. However the preference profile V has the majority cycle $a > c > b > a$.

Observation 7 *There exists a preference profile of weak orders that is single-peaked₃ consistent and does not have a transitive majority relation.*

2.2 Consecutive-Ones Matrices

All of our polynomial-time results are results of transformations to the problem of determining if a 0-1 matrix has the property of consecutive ones in rows that we define below.

Definition 8 *A 0-1 matrix M has the property of consecutive ones in rows if the columns of M can be permuted so that in each row all of the 1's are consecutive.*

The problem of determining if a 0-1 matrix has the property of consecutive ones in rows was shown to be in P by Fulkerson and Gross (1965). Booth and Lueker (1976) improved on this result by finding a linear-time algorithm through the

¹ V is listed in a table of inadmissible preferences for single-peakedness in Chapter 9 of Fishburn (1973).

development of the novel PQ-tree data structure, which contains all possible permutations of the columns that have the property of consecutive ones in rows.

The following three sections consist of our results and they are structured as follows. We examine each variant of single-peakedness starting with the weakest restriction and ending with the strongest. When we examine each restriction we present a formal definition of the variant of single-peakedness and the transformation to the problem of determining if a 0-1 matrix has consecutive ones in rows.

3 Existentially Single-Peaked Consistency

We start with single-peaked₃ preferences since it is the most general of the three variants mentioned in Section 2. The construction and corresponding proof will be the basis for showing that single-peaked and single-plateaued consistency for weak orders are each also in P.

Given an axis A and a preference order v , if v is single-peaked₃ with respect to A then v cannot have strictly decreasing and then strictly increasing preference with respect to A . Following the terminology used by Lackner (2014), we refer to this as a v-valley.

Definition 9 *A preference order v over a candidate set C contains a v-valley with respect to an axis A if and only if there exist distinct candidates $a, b, c \in C$ such that $a < b < c$ in A and $a > b$ and $c > b$ in v .*

Using the v-valley substructure we can state the following.

Lemma 10 (Lackner 2013) *Let V be a preference profile of weak orders. V is single-peaked₃ with respect to an axis A if and only if no preference order $v \in V$ contains a v-valley with respect to A .*

Lemma 10 will simplify our argument used in the proof of Theorem 13.

To construct a matrix from a preference profile of weak orders, we apply essentially the same transformation as Bartholdi and Trick (1986) for total orders (see Example 12).

Construction 11 *For each $v_i \in V$ construct a $(\|C\| - 1) \times \|C\|$ matrix X_i . Each column of X_i corresponds to a candidate in C . For each candidate $c \in C$ let k be the number of candidates that are ranked strictly above c in v_i and let the corresponding column in matrix X_i contain $(\|C\| - k - 1)$ 1's starting at row zero, with the remaining entries filled with 0's. All $\|V\|$ of the matrices are then row-wise concatenated to yield the $(\|V\| \times (\|C\| - 1)) \times \|C\|$ matrix X .*

The main differences in our construction are the determination of the number of 1's in each column, that each column is not guaranteed to have a distinct number of 1's, and that we have one fewer row in each of the individual preference matrices. In the construction used by Bartholdi and Trick (1986), given a preference order v over a set of candidates C , for all candidates $a, b \in C$, $a > b$ in v if and only if the number of 1's in the column corresponding to a is greater than the number of 1's in the column corresponding to b in v 's corresponding individual preference matrix. Notice that this still holds for our construction even though we determine the number of 1's in each column differently.

It is easy to see that this change will not affect the presence or absence of the restricted sequence $\dots 1 \dots 0 \dots 1 \dots$ in a row.

Some of the results stated in Lackner (2014) use a *guiding order* or an implicitly specified total order. Given a preference profile V , a guiding order can be constructed iteratively as follows. If there exists a $v \in V$ such that the last ranked candidate in v is not ranked indifferently with any other candidate, then that candidate is appended to the top of the guiding order. This is then repeated on the preference profile restricted to the candidates not yet added to the guiding order until either the guiding order is a total order or there is no $v \in V$ with a unique last ranked candidate, the case where no guiding order exists (Lackner 2014). If a given preference profile is single-peaked₃ then it remains single-peaked₃ if a guiding order is added as an additional preference order (Lackner 2014). Lackner (2014) utilizes guiding orders for his polynomial-time algorithms determining the single-peaked₃ consistency for weak and local weak orders. Below we show how Construction 11 is applied.

Example 12 Consider the preference profile V that consists of the preference orders v_1 and v_2 . Let the preference order v_1 be $(a \sim c > b > e \sim d)$ and the preference order v_2 be $(a > b > c > e \sim d)$. Notice that the preference profile V does not contain a guiding order, which is required by the polynomial-time algorithm for weak orders found in Lackner (2014).

$$X_1 = \begin{bmatrix} a & b & c & d & e \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} a & b & c & d & e \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We row-wise concatenate the matrices X_1 and X_2 to construct the matrix X .

$$X = \begin{bmatrix} a & b & c & d & e \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad X' = \begin{bmatrix} b & a & c & d & e \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Next, we permute the columns of X so that in each row all of the 1's are consecutive to yield X' . Observe that V is single-peaked₃ with respect to $b < a < c < d < e$, the ordering of the columns of X' , as its axis.

We now show that single-peaked₃ consistency for weak orders and the problem of determining if a 0-1 matrix has consecutive ones in rows are equivalent using Construction 11 and Lemma 10.

Theorem 13 A preference profile V of weak orders is single-peaked₃ consistent if and only if there exists a permutation of the candidates such that X , constructed using Construction 11, has the property of consecutive ones in rows.

Proof. Let V be a preference profile of weak orders. Essentially the same argument as used by Bartholdi and Trick (1986) holds.

If V is single-peaked₃ then there exists an axis A such that no $v \in V$ contains a v-valley with respect to A . When the columns of the matrix X are permuted to correspond to the axis A no row will contain the sequence $\dots 1 \dots 0 \dots 1 \dots$ since this would correspond to a preference order that decreases and then increases along the axis A (a v-valley). Therefore X has the property of consecutive ones in rows.

If V is not single-peaked₃ then for every possible axis there exists a $v \in V$ such that v contains a v-valley with respect to that axis. So every permutation of the columns of X will correspond to an axis where some preference order has a v-valley. As stated in the other direction, a v-valley corresponds to a row containing the sequence $\dots 1 \dots 0 \dots 1 \dots$ so clearly X does not have the property of consecutive ones in rows.

The only difference from the argument used by Bartholdi and Trick (1986) for total orders is that in our case the preference orders can remain constant at the peak and at points on either side of the peak. The same argument still applies since by Lemma 10 the absence of v-valleys with respect to an axis is equivalent to a profile of weak orders being single-peaked₃ with respect to that axis. \square

As a result of the proof of Theorem 13 the PQ-tree algorithm by Booth and Lueker (1976) can be applied to determine the single-peaked consistency for a given preference profile of weak orders in polynomial time.

Theorem 14 Determining if a given preference profile V of weak orders is single-peaked₃ is in P.

4 Single-Plateaued Consistency

Single-plateaued preferences are much more restrictive than single-peaked₃ preferences since they are essentially single-peaked except that they allow each preference order to have multiple top preferred candidates (Black 1958, Chapter 5).

Since a preference order must be *strictly* increasing and then *strictly* decreasing with respect to an axis A (excluding its most preferred candidates) we can again use the v-valley substructure. However we will need another substructure to prevent two candidates that are indifferent in a voter's preference order from appearing on the same side of their peak.

Definition 15 A preference order v over a candidate set C contains a nonpeak plateau with respect to A if and only if there exist distinct candidates $a, b, c, \in C$ such that $a < b < c$ in A and either $a > b \sim c$ or $a \sim b < c$ in v .

We use the v-valley and nonpeak plateau substructures to state the following lemma.

Lemma 16 Let V be a preference profile of weak orders. V is single-plateaued with respect to an axis A if and only if no preference order $v \in V$ contains a v-valley or a nonpeak plateau with respect to A .

Since the nonpeak plateau substructure is needed in addition to the v-valley substructure we need to extend Construction 11 so that if a preference order contains a nonpeak

plateau with respect to an axis A , then when the columns of its preference matrix are permuted according to A the matrix will contain a row with the sequence $\dots 1 \dots 0 \dots 1 \dots$.

Construction 17 For each $v_i \in V$ construct a $(\|C\| - 1) \times \|C\|$ matrix X_i . Each column of X_i corresponds to a candidate in C . For each candidate $c \in C$ let k be the number of candidates that are ranked strictly above c in v_i and let the corresponding column in matrix X_i contain $(\|C\| - k - 1)$ 1's starting at row zero, with the remaining entries filled with 0's (as in Construction 11).

If there exist two distinct candidates $a, b \in C$ such that $a \sim b$ in v_i , they are not the most preferred candidates in v_i , and there is no candidate $c \in C - \{a, b\}$ such that v_i is indifferent among a, b , and c , then append three additional rows below row zero of X_i where the column corresponding to a is $[0 \ 1 \ 1]'$, the column corresponding to b is $[1 \ 1 \ 0]'$, each column corresponding to a candidate ranked strictly greater than a and b is $[1 \ 1 \ 1]'$, and each column corresponding to a remaining candidate is $[0 \ 0 \ 0]'$.

If there exist three distinct candidates $a, b, c \in C$ that are ranked indifferent by v_i and they are not the most preferred candidates in v_i , then output "no solution" since there is no ordering where the preference profile containing v_i is single-plateaued.

After constructing an X_i matrix for each v_i all $\|V\|$ of the matrices are row-wise concatenated to yield a matrix X .

Below is an example of Construction 17 applied to a preference profile of weak orders that is single-plateaued.

Example 18 We consider the same preference profile as in Example 12 and we bold the additional rows in this example. Let the preference order v_1 be $(a \sim c > b > e \sim d)$ and the preference order v_2 be $(a > b > c > e \sim d)$.

$$X_1 = \begin{bmatrix} a & b & c & d & e \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} a & b & c & d & e \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

We row-wise concatenate the matrices X_1 and X_2 to construct the matrix X .

$$X = \begin{bmatrix} a & b & c & d & e \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad X' = \begin{bmatrix} e & b & a & c & d \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Next, we permute the columns of X such each in each row all of the ones are consecutive to yield X' . Observe that V is single-plateaued with respect to this new ordering $e < b < a < c < d$ as its axis. Also notice that an axis containing d and e adjacent to each other (as seen in Example 12) would not correspond to an ordering of the columns of X that would have the property of consecutive ones in rows due to the additional rows added because of the extensions made to Construction 11 in Construction 17.

Construction 11 ensures that no preference order contains a v-valley and the extensions in Construction 17 ensures that no preference order contains a nonpeak plateau. Therefore the proof of the following theorem uses a similar argument to the proof of Theorem 13 but uses Lemma 16. Now the presence of v-valleys or nonpeak plateaus (not just v-valleys) is equivalent to a row containing the sequence $\dots 1 \dots 0 \dots 1 \dots$.

Theorem 19 A preference profile V of weak orders is single-plateaued consistent if and only if there exists a permutation of the candidates such that X , constructed using Construction 17, has the property of consecutive ones in rows.

As a result of the proof of Theorem 19 the PQ-tree algorithm by Booth and Lueker (1976) can be applied to determine single-peaked consistency for weak orders in polynomial time.

Theorem 20 Determining if a given preference profile V of weak orders is single-plateaued is in P.

5 Single-Peaked Consistency

We now present the strongest domain restriction on weak orders that we examine, single-peaked preferences. A preference order is single-peaked with respect to an axis A if it is *strictly* increasing to a peak and then *strictly* decreasing with respect A . So we again use the v-valley substructure. However, even if no preference order has a v-valley with respect to a given axis it may not be single-peaked because it is indifferent between two candidates on the same side of its peak or has more than one most preferred candidate. The first condition can be handled with the nonpeak plateau substructure used in Section 4, but the second condition requires us to view any plateau as a forbidden substructure, not just nonpeak plateaus.

Definition 21 A preference order v over a candidate set C contains a plateau with respect to an axis A if and only if there exist distinct candidates $a, b \in C$ such that a and b are adjacent in A and $a \sim b$ in v .

We can now use the plateau substructure and the v-valley substructure to state the following lemma.

Lemma 22 Let V be a preference profile of weak orders. V is single-peaked with respect to an axis A if and only if no preference order $v \in V$ contains a v-valley or a plateau with respect to A .

We now extend Construction 17 so that if a preference order contains a plateau with respect to an axis A , then when the columns of its preference matrix are permuted according to A the matrix will contain the sequence $\dots 1 \dots 0 \dots 1 \dots$.

Construction 23 Follow Construction 17 except add the following additional condition while constructing each X_i matrix from a preference order v_i .

If there exist two distinct candidates $a, b \in C$ such that $a \sim b$ in v_i and they are the most preferred candidates in v_i , then output “no solution” since there is no ordering where the preference profile containing v_i is single-peaked.

When each preference order in a given preference profile has a unique most preferred candidate and the preference profile is single-plateaued it is clearly also single-peaked. Construction 23 simply adds a condition that outputs “no solution” when a preference order has more than one most preferred candidate to Construction 17. Therefore the proof of the following theorem follows directly from the proof of Theorem 19.

Theorem 24 A preference profile V of weak orders is single-peaked consistent if and only if there exists a permutation of the candidates such that X , constructed using Construction 23 has the property of consecutive ones in rows

As a result of the proof of Theorem 24 the PQ-tree algorithm by Booth and Lueker (1976) can be applied to determine single-peaked consistency for weak orders in polynomial time.

Theorem 25 Determining if a given preference profile V of weak orders is single-peaked is in P.

6 Related Work

Determining the single-peaked consistency for a preference profile of total orders was first shown to be in P by Bartholdi and Trick (1986) through a transformation to the problem of determining if a 0-1 matrix has the property of consecutive 1’s in rows. Later, both Escoffier et al. (2008) and Doignon and Falmagne (1994) independently found faster direct algorithms for the single-peaked consistency for total orders.

Single-peaked and single-plateaued preferences are closely related domain restrictions. Barberà (2007) discusses how the changes in the amount of indifference permitted in different variants of single-peaked preferences impact their properties. Fishburn (1973) shows how indifference and partial orders affect single-peaked preferences and presents a definition for single-peaked preferences on strict partial orders and Fishburn states that his definition of single-peaked preferences on weak orders is equivalent to the definition of single-plateaued preferences in Chapter 5 of Black (1958).

One approach to dealing with partial preferences is to assume that voters have an underlying total preference order (Konczak and Lang 2005) and consider possible extensions of their preferences to total orders. This is the approach taken by Lackner (2014) for the existential model of single-peaked consistency. The existential approach is easily applied to other domain restrictions and has already been applied to top monotonicity (Barberà and Moreno 2011) for partial orders (Aziz 2014). In this case a preference profile of partial orders is top monotonic if there exists an extension to a profile of weak orders that are top monotonic since top monotonicity is defined on weak orders.

7 Conclusions and Open Problems

The three different variants of single-peakedness for weak orders, each permitting different amounts of indifference, were presented and each of their corresponding consistency problems for weak orders were shown to be in P. Since they were shown in P through transformations to the problem of determining if a 0-1 matrix has the property of consecutive ones in rows we were able to apply the PQ-tree algorithm by Booth and Lueker (1976). Using this algorithm we can actually go further than just determining the consistency problem for each of these variants and actually find *all* consistent axes. Transformations to the same consecutive-ones matrix problem were applied to show single-peaked consistency for total orders to be in P and also to show an alternative algorithm for single-crossing consistency for total orders (Bredereck, Chen, and Woeginger 2013). An interesting open problem is how the consecutive-ones matrix problem relates to other domain restrictions and what benefits there are to having all consistent axes for a given preference profile.

Single-peaked preferences are studied because they are a simply stated domain restriction with nice properties that are likely to occur in real-world settings. However, experimental study has found that in real-world settings voters are often not single-peaked (Mattei, Forshee, and Goldsmith 2012). In Mattei et al. (2012) the preferences of the voters were extended to total orders and it would be interesting to see if the presence of single-peaked preferences would be more likely if the original weak orders were preserved.

Elections that contain voters that have nearly single-peaked preferences have beneficial properties (Faliszewski, Hemaspaandra, and Hemaspaandra 2014). There are several different types of nearly single-peakedness and detecting them is an interesting computational problem (Erdélyi, Lackner, and Pfandler 2013) and it would be interesting to see how preference profiles of weak orders impact the complexity of nearly single-peakedness or, as mentioned in Lackner (2014), nearly single-peakedness in the existential model.

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References

- Aziz, H. 2014. Testing top monotonicity. Technical Report arXiv:1403.7625 [cs.GT], arXiv.org.
- Barberà, S., and Moreno, B. 2011. Top monotonicity: A common root for single peakedness, single crossing and the median voter result. *Games and Economic Behavior* 73(2):345–359.
- Barberà, S. 2007. Indifferences and domain restrictions. *Analyse & Kritik* 29(2):146–162.
- Bartholdi, III, J., and Trick, M. 1986. Stable matching with preferences derived from a psychological model. *Operations Research Letters* 5(4):165–169.

- Black, D. 1948. On the rationale of group decision-making. *Journal of Political Economy* 56(1):23–34.
- Black, D. 1958. *The Theory of Committees and Elections*. Cambridge University Press.
- Booth, K., and Lueker, G. 1976. Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *Journal of Computer and System Sciences* 13(3):335–379.
- Brandt, F.; Brill, M.; Hemaspaandra, E.; and Hemaspaandra, L. A. 2010. Bypassing combinatorial protections: Polynomial-time algorithms for single-peaked electorates. In *Proceedings of the 24th AAAI Conference on Artificial Intelligence (AAAI 2010)*, 715–722.
- Bredereck, R.; Chen, J.; and Woeginger, G. 2013. A characterization of the single-crossing domain. *Social Choice and Welfare* 41(4):989–998.
- Doignon, J.-P., and Falmagne, J.-C. 1994. A polynomial time algorithm for unidimensional unfolding representations. *Journal of Algorithms* 16(2):218–233.
- Emerson, P. 2013. The original Borda count and partial voting. *Social Choice and Welfare* 40(2):352–358.
- Erdélyi, G.; Lackner, M.; and Pfandler, A. 2013. Computational aspects of nearly single-peaked electorates. In *Proceedings of the 27th AAAI Conference on Artificial Intelligence (AAAI 2013)*, 283–289.
- Escoffier, B.; Lang, J.; and Öztürk, M. 2008. Single-peaked consistency and its complexity. In *Proceedings of the 18th European Conference on Artificial Intelligence (ECAI 2008)*, 366–370.
- Faliszewski, P.; Hemaspaandra, E.; Hemaspaandra, L. A.; and Rothe, J. 2011. The shield that never was: Societies with single-peaked preferences are more open to manipulation and control. *Information and Computation* 209:89–107.
- Faliszewski, P.; Hemaspaandra, E.; and Hemaspaandra, L. A. 2014. The complexity of manipulative attacks in nearly single-peaked electorates. *Journal of Artificial Intelligence Research* 207:69–99.
- Fishburn, P. 1973. *The Theory of Social Choice*, volume 264. Princeton University Press.
- Fulkerson, D., and Gross, O. 1965. Incidence matrices and interval graphs. *Pacific Journal of Math* 15(3):835–855.
- Hemaspaandra, E.; Hemaspaandra, L. A.; and Rothe, J. 1997. Exact analysis of Dodgson elections: Lewis Carroll’s 1876 voting system is complete for parallel access to NP. *Journal of the ACM* 44(6):806–825.
- Hemaspaandra, E.; Spakowski, H.; and Vogel, J. 2005. The complexity of Kemeny elections. *Theoretical Computer Science* 349(3):382–391.
- Kemeny, J. 1959. Mathematics without numbers. *Daedalus* 88:577–591.
- Konczak, K., and Lang, J. 2005. Voting procedures with incomplete preferences. In *Proceedings of the 1st Multidisciplinary Workshop on Advances in Preference Handling (M-PREF 2005)*, 124–129.
- Lackner, M. 2013. Incomplete preferences in single-peaked electorates. In *Proceedings of the 7th Multidisciplinary Workshop on Advances in Preference Handling (M-PREF 2013)*.
- Lackner, M. 2014. Incomplete preferences in single-peaked electorates. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI 2014)*.
- Mattei, N., and Walsh, T. 2013. PREFLIB: A library for preferences. In *Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT 2013)*. Springer. 259–270.
- Mattei, N.; Forshee, J.; and Goldsmith, J. 2012. An empirical study of voting rules and manipulation with large datasets. In *Proceedings (Workshop Notes) of the 4th International Workshop on Computational Social Choice*.
- Narodytska, N., and Walsh, T. 2014. The computational impact of partial votes on strategic voting. In *Proceedings of the 21st European Conference on Artificial Intelligence (ECAI 2014)*.